

ON THE SPACE OF ORIENTED AFFINE LINES IN \mathbb{R}^3

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ABSTRACT. We introduce a local coordinate description for the correspondence between the space of oriented affine lines in Euclidean \mathbb{R}^3 and the tangent bundle to the 2-sphere. These can be utilised to give canonical coordinates on surfaces in \mathbb{R}^3 , as we illustrate with a number of explicit examples.

The correspondence between oriented affine lines in \mathbb{R}^3 and the tangent bundle to the 2-sphere has a long history and has been used in various contexts. In particular, it has been used in the construction of minimal surfaces [2], solutions to the wave equation [3] and the monopole equation [1].

The Euclidean group of rotations and translations acts upon the space of oriented lines \mathcal{L} and in this paper we freeze out this group action by introducing a particular set of coordinates on \mathcal{L} . Our aim is to provide a local coordinate representation for the correspondence, thereby making it accessible to further applications.

One application is the construction of canonical coordinates on surfaces S in \mathbb{R}^3 which come from the description of the normal lines of S as local sections of the tangent bundle of the 2-sphere. We illustrate this explicitly by considering the ellipsoid and the symmetric torus.

Definition 1. Let \mathcal{L} be the set of oriented (affine) lines in Euclidean \mathbb{R}^3 .

Definition 2. Let $\Phi : TS^2 \rightarrow \mathcal{L}$ be the map that identifies \mathcal{L} with the tangent bundle to the unit 2-sphere in Euclidean \mathbb{R}^3 , by parallel translation. This bijection gives \mathcal{L} the structure of a differentiable 4-manifold.

Let (ξ, η) be holomorphic coordinates on TS^2 , where ξ is obtained by stereographic projection from the south pole onto the plane through the equator, and we identify (ξ, η) with the vector

$$\eta \frac{\partial}{\partial \xi} + \bar{\eta} \frac{\partial}{\partial \bar{\xi}} \in T_{\xi} S^2.$$

Theorem 1. The map Φ takes $(\xi, \eta) \in TS^2$ to the oriented line given by

$$z = \frac{2(\eta - \bar{\eta}\xi^2) + 2\xi(1 + \xi\bar{\xi})r}{(1 + \xi\bar{\xi})^2} \quad (0.1)$$

$$t = \frac{-2(\eta\bar{\xi} + \bar{\eta}\xi) + (1 - \xi^2\bar{\xi}^2)r}{(1 + \xi\bar{\xi})^2}, \quad (0.2)$$

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where $z = x^1 + ix^2$, $t = x^3$, (x^1, x^2, x^3) are Euclidean coordinates on $\mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R}$ and r is an affine parameter along the line such that $r = 0$ is the point on the line that lies closest to the origin.

Proof. Stereographic projection from the south pole gives a map from \mathbb{C} to \mathbb{R}^3 by

$$z = \frac{2\xi}{1 + \xi\bar{\xi}}, \quad t = \frac{1 - \xi\bar{\xi}}{1 + \xi\bar{\xi}}. \quad (0.3)$$

The derivative of this map gives

$$\frac{\partial}{\partial \xi} = \frac{2}{(1 + \xi\bar{\xi})^2} \frac{\partial}{\partial z} - \frac{2\bar{\xi}^2}{(1 + \xi\bar{\xi})^2} \frac{\partial}{\partial \bar{z}} - \frac{2\bar{\xi}}{(1 + \xi\bar{\xi})^2} \frac{\partial}{\partial t},$$

and similarly for its conjugate.

Thus

$$\eta \frac{\partial}{\partial \xi} + \bar{\eta} \frac{\partial}{\partial \bar{\xi}} = \frac{2(\eta - \bar{\eta}\xi^2)}{(1 + \xi\bar{\xi})^2} \frac{\partial}{\partial z} + \frac{2(\bar{\eta} - \eta\bar{\xi}^2)}{(1 + \xi\bar{\xi})^2} \frac{\partial}{\partial \bar{z}} - \frac{2(\eta\bar{\xi} + \bar{\eta}\xi)}{(1 + \xi\bar{\xi})^2} \frac{\partial}{\partial t}. \quad (0.4)$$

Consider the line in \mathbb{R}^3 given by equations (0.1) and (0.2). The direction of this line is

$$\frac{2\xi}{1 + \xi\bar{\xi}} \frac{\partial}{\partial z} + \frac{2\bar{\xi}}{1 + \xi\bar{\xi}} \frac{\partial}{\partial \bar{z}} + \frac{1 - \xi\bar{\xi}}{1 + \xi\bar{\xi}} \frac{\partial}{\partial t}.$$

When this unit vector is translated to the origin, it ends at the point $\xi \in S^2$ (cf. equation (0.3))

The fixed vector determining the line is seen to be (0.4), and, using the fact that the Euclidean inner product of the basis vectors is

$$\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) = \frac{1}{2} \quad \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = 1$$

$$\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) = \left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial \bar{z}} \right) = \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial t} \right) = \left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial t} \right) = 0,$$

we compute that the line is orthogonal to the fixed vector given by (0.4). Thus r is an affine parameter along the line such that $r = 0$ is the point on the line that lies closest to the origin, and the proof is completed. \square

Consider the map $\Psi : \mathcal{L} \times \mathbb{R} \rightarrow \mathbb{R}^3$ which takes a line and a number r to a point on the line which is a parameter distance r from the point on the line closest to the origin.

Proposition 1. Ψ^{-1} takes a point $(z, t) \in \mathbb{R}^3$ to a sphere in $\mathcal{L} \times \mathbb{R}$, the oriented lines containing the point:

$$\eta = \frac{1}{2}(z - 2t\xi - \bar{z}\xi^2) \quad r = \frac{\bar{\xi}z + \xi\bar{z} + (1 - \xi\bar{\xi})t}{1 + \xi\bar{\xi}}.$$

Proof. This comes from solving equations (0.1) and (0.2) for η and r .

Alternatively, the second equation can be proved by finding the point p on the line with direction ξ through $(z, t) \in \mathbb{R}^3$ which minimises the distance to the origin. Then $r^2 = |(z, t)|^2 - |p|^2$, which gives the above expression for r . \square

By throwing away the r information, the above formula gives the holomorphic sphere of lines through a given point $(z, t) \in \mathbb{R}^3$, as described in [1]. These are a 3-parameter family of global sections of TS^2 and the associated line congruence in \mathbb{R}^3 is normal to round spheres about the given point.

More generally any oriented surface S in \mathbb{R}^3 gives rise to a surface $\Sigma \subset \mathcal{L}$ through its normal line congruence. Such a Σ will, in general, not be holomorphic, nor be given by global sections of the bundle. However, locally, a surface can often be given by local non-holomorphic sections and the following examples illustrate this for two well-known surfaces.

The examples can be verified by substitution in equations (0.1) and (0.2) and then checking that the resulting surface, parameterised by its normal direction coordinate ξ , is indeed the one claimed.

Example 1. The triaxial ellipsoid with semi-axes a_1 , a_2 and a_3 can be covered by coordinates ξ via

$$\eta = \frac{a_1(\xi + \bar{\xi})(1 - \xi^2) + a_2(\xi - \bar{\xi})(1 + \xi^2) - 2a_3\xi(1 - \xi\bar{\xi})}{2\sqrt{a_1(\xi + \bar{\xi})^2 - a_2(\xi - \bar{\xi})^2 + a_3(1 - \xi\bar{\xi})^2}}$$

$$r = \sqrt{a_1\left(\frac{\xi + \bar{\xi}}{1 + \xi\bar{\xi}}\right)^2 - a_2\left(\frac{\xi - \bar{\xi}}{1 + \xi\bar{\xi}}\right)^2 + a_3\left(\frac{1 - \xi\bar{\xi}}{1 + \xi\bar{\xi}}\right)^2}.$$

These coordinates extend to $\xi \rightarrow \infty$ and so this is an example of a global non-holomorphic section of $\pi : TS^2 \rightarrow S^2$.

Example 2. The rotationally symmetric torus of radii a and b is given by

$$\eta = \pm \frac{a}{2} \sqrt{\frac{\xi}{\bar{\xi}}} (1 - \xi\bar{\xi})$$

$$r = b \pm \frac{2a\sqrt{\xi\bar{\xi}}}{1 + \xi\bar{\xi}}.$$

This describes the torus as a double cover of the 2-sphere, branched at the north and south poles.

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